Usage of Orthogonal Polynomials at Calculation of Transfer Processes in Electric Circuits with Variable Parameters Using Differential Transformations

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Abstract. The paper presents a new method for characterization of circuit with variable parameters in the form of extended series by using the differential-Taylor transformations. Also ways of solution increasing precision are considered at reduction of number of series terms by using a conversion from degree in a base of orthogonal polynomials of Lagger, Legandre, and Chebyshev's 1 type and displaced polynomials.

I. Introduction

In electrical engineering, automation, mechanics and other fields the integration transformations have been wide-spreaded. The initial differential equations are transformed to the more simply algebraic equations by this transformation. The special popularity the integral transformations have got at a research of linear systems with constant parameters. However using this operators does not give appreciable advantages on comparison with the known numerical methods at the analysis of systems with variable and nonlinear parameters.

The differential transformations (differential-Taylor, or DT-transformation) of functions [1] and equations are used in tasks where a solution on a given modification interval of independent variable, for example, time, effectively can be represented by segment of Taylor series or by local series:

$$x(t)=C_0+\frac{t}{H}C_1+\left(\frac{t}{H}\right)^2C_2+...=\sum_{k=0}^{k=\infty}\left(\frac{t}{H}\right)^kC_k, \qquad (1)$$

The DT-transformations allow to transform the initial differential equations both linear and nonlinear into algebraic equations. The transformation looks as follows:

$$X(k)=\frac{H^k}{k!}\left[\frac{d^kx(t)}{dt^k}\right]_{t=0} \quad \equiv \quad x(t)=\sum_{k=0}^{\infty}\left(\frac{t}{H}\right)^kX(k), \qquad (2)$$

where at the left there is a direct transformation of original x(t) into image X(k), and on the right – reconversion X(k) to x(t). Values of the function X(k) at concrete values of argument k are named discrets (X(0) – zero discrete, X(1) – first discrete, etc.). Thus given mathematical model of system in the form of integral-differential equations can be transformed to so-called spectral model, by replacement of functions, included to mathematical model, and operations with them by differential spectrum. The spectral model usually represents a system of recurrent type of final equations, from which the discrets of differential spectrum of unknown quantities can be easily found. At last stage the unknown quantities of function as segments of ascending extended series Taylor (1) basing on reconversions (2) are founded practically without any evaluations, and, the more discrets we shall have, the more precisely we shall restore unknown functions.

II. The problem

For example, let's take a circuit with variable parameter consisting of series-tuned circuit with a resistor, which impedance depend of time, as shown on the figure 1.

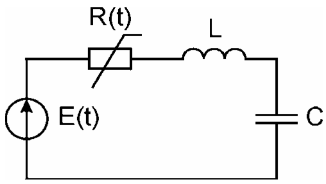

Fig. 1. Series-tuned circuit with a nonlinear resistor.

It is necessary to define voltage on the capacitor $U_C(t)$, under conditions $R(t)=R_0\cdot e^{-a\cdot t}$, R_0=10 Ohm, a=1 sec^{-1}, L=2 Hn, C=10000 μF, $E(t)=E_0$, at t>0, E(0)=0, E_0=1 V, the capacitor and the inductor are in the unloaded state. Let's note a set of equations of our circuit:

$$\begin{cases} E(t)=I(t)\cdot R(t)+U_L(t)+U_C(t), \\ E(t)=\begin{cases} 0, & t\le 0, \\ E_0, & t>0. \end{cases} \\ R(t)=R_0\cdot e^{-\alpha\cdot t}, \\ U_L(t)=L\dfrac{dI(t)}{dt}, \\ U_C(t)=\dfrac{1}{C}\int_0^t I(t)dt. \end{cases} \qquad (3)$$

From (3) an expression for the sum of voltages in the circuit is obtained:

$$E(t)=I(t)\cdot R_0\cdot e^{-\alpha\cdot t}+L\frac{dI(t)}{dt}+\frac{1}{C}\int_0^t I(t)dt\,. \qquad (4)$$

Where the current of the circuit can be expressed from capacity voltage:

$$I(t)=C\frac{dU_C(t)}{dt} \qquad (5)$$

Let's take advantage (5) and we shall write the equation (4) for voltage in the capacitor:

$$E(t)=L\cdot C\frac{d^2U_C(t)}{dt}+R_0\cdot e^{-\alpha\cdot t}\cdot C\frac{dU_C(t)}{dt}+U_C(t) \qquad (6)$$

Let's take advantage of the transformations table [1], and we get the system (3) in a spectral form:

$$\begin{cases} E(t): & E(k) = E_0 \cdot \delta(k) \\ & \delta(\mathrm{k}) = \begin{cases} 1, & k = 0 \\ 0, & k \geq 1 \end{cases} \\ U_C(t): & X(k) \\ \dfrac{dU_C(t)}{dt}: & \dfrac{k+1}{H} X(k+1) \\ \dfrac{d^2U_C(t)}{dt}: & \dfrac{(k+2)!}{k! \cdot H^2} X(k+2) \\ e^{-a \cdot t}: & \dfrac{(-a \cdot H)^k}{k!} \end{cases} \qquad (7)$$

Using following properties:

$$\begin{aligned} &c \cdot x(t): \quad x \cdot X(k), \\ &x(t) \pm y(t): \quad X(k) \pm Y(k), \\ &x(t) \cdot y(t): \quad \sum_{l=0}^{l=k} X(k-l) Y(l), \end{aligned} \qquad (8)$$

we obtain the differential equation (6) in the spectral form:

$$\begin{aligned} &E_0 \cdot \delta(k) = L \cdot C \frac{(k+2)!}{k! \cdot H^2} X(k+2) + \\ &+ \sum_{l=0}^{l=k} \left(\frac{(-a \cdot H)^{k-l}}{(k-l)!} X(l+1) \frac{l+1}{H} \right) + X(k), \end{aligned} \qquad (9)$$

Thus, at a substitution in (9) k=0,1,2, …, we shall take a recursion formula for an calculating discrets X(k):

$$\begin{aligned} &X(k+2) = \frac{H^2}{L \cdot C \cdot (k+1) \cdot (k+2)} \times \\ &\times \left[E_0 \cdot \delta(k) - R_0 \cdot C \cdot \sum_{l=0}^{l=k} \left(\frac{(-a \cdot H)^{k-l}}{(k-l)!} X(l+1) \frac{l+1}{H} \right) - X(k) \right] \end{aligned} \qquad (10)$$

Let's calculate first four discrets, outgoing from the initial conditions $X(0) = X(1) = U_C(0) = 0$:

$$k = 0, \quad X(2) = \frac{E_0 \cdot H}{2 \cdot L \cdot C},$$

$$k = 1, \quad X(3) = -\frac{E_0 \cdot R_0 \cdot H^3}{6 \cdot L^2 \cdot C},$$

...

Using the obtained expression for discrets X(k) calculation, we can restore the required function $U_C(t)$ with the help of the reconversion (2) and we shall have a solution of the system (3) in an analytical kind. At the known entry conditions we calculate the numerical values of the discrets X(k) (fig. 2, DT discrets) and restore the signal $U_C(t)$ (fig. 3). Thus, we get the solution of the system (3) as extended series.

There are some problems in the discrets calculating by essential lack of the method when find solution of actual tasks. The more discrets will be calculates the more precise will be obtained solution, as it will represent more precise Taylor series.

At the figure 4 a mean square deviation (MSD) depending on amount of discrets, taken at signal recovery is represented. From the figure 3 it is visible that 16 discrets is obvious insufficiently and MSD makes 17% (Fig. 4), and the graph (Fig. 3, the dash line) hardly differs from actual (Fig. 3, the solid line). If to take large amount of discrets 21-31, the restored signal completely coincide with real signal, and the error MSD makes 3.6…0.066% accordingly.

So that to find a solution of the system on the given time interval with a greater precision, it is necessary to calculate more number of discrets, accordingly an obtained extended series will more number of terms, that is not always convenient.

On the other hand an evaluation of large amount of discrets can appear inexpediently, as will not increase a precision of the obtained solution. Then there is a problem, how much discrets necessary to evaluate to get the desired precision of the solution. In a degree base it is impossible uniquely to say, in what moment it is necessary to stop evaluations, as it is impossible with a confidence to say, that at reaching demanded small value of discrets the sum following discarded discrets will be minute.

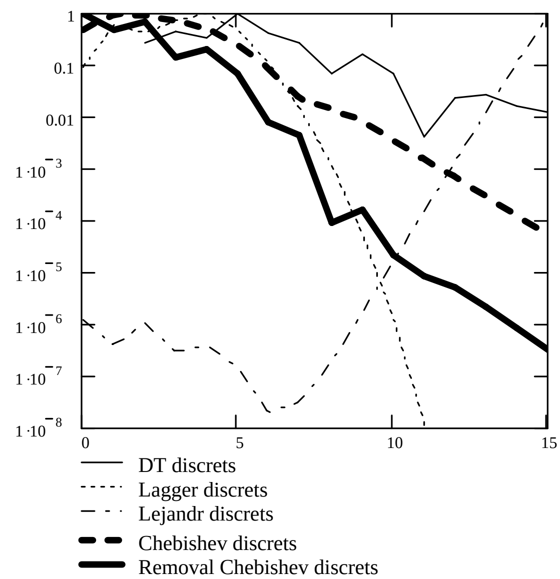

Fig. 2. Discrets of spectrum.

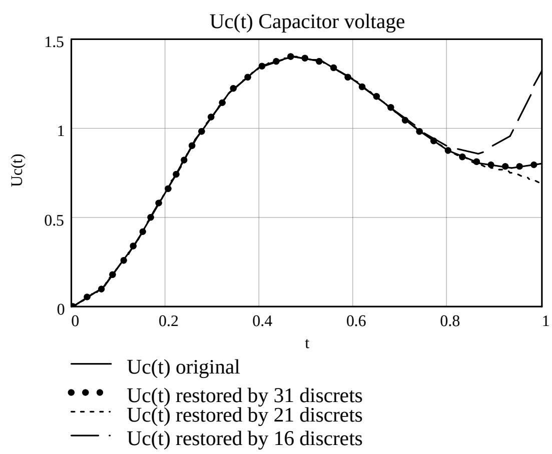

Fig. 3. Restored signal.

For cutting number of series terms and increasing precision of restoring it is possible to take advantage of transition from degree base in base of orthogonal polynomials. That is we shall search for a solution not by the way (2), but this way:

$$x(t)=\sum_{i=0}^{\infty}C(i)T_i(t) \qquad (11)$$

where C(i) are discrets of a spectrum in a base of orthogonal polynomials, and $T_i(t)$ are numerical value of polynomial in given base.

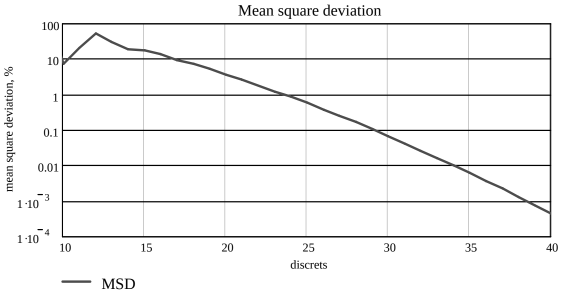

Fig. 4. Mean square deviation depending on discrets amount.

III. Transition in base of Lagger polynomials

For converting discrets of differential spectrum (DT-discrets) in base of Lagger polynomials (Lag-discrets), it is necessary beforehand to have a definite amount of DT-discrets, as for calculation of each coefficient C(i) the maximum number of discrets of the initial base are used:

$$C(i)=\sum_{k=0}^{k=km}\left[X(k)\cdot\sum_{j=0}^{j=i}\left(\frac{(-1)^j\cdot(k+j)!}{(j!)^2\cdot(i-j)!}\right)\right] \qquad (12)$$

where *km* is maximum known number of DT-discrets.

Value of Lagger polynomials $T_i(t)$ are evaluated under the following formula:

$$T_i(t)=\sum_{k=0}^{k=i}\left(\frac{(-1)^k\cdot(i!)^2}{(k!)^2\cdot(i-k)!}\cdot t^k\right) \qquad (13)$$

At evaluation of discrets by the formula (12) because of availability of high factorials the error is accumulated, which one, in practice, does not allow to calculate discrets above than 11-13 order. Therefore at calculation only 12 discrets of a differential spectrum were used, that it is obviously not enough for precise signal restoring and, accordingly, obtained signal will hardly differ from true. But we shall check up quality of transition in base of Lagger polynomials for the obtained signal. As it is visible (Fig. 2, Lagger discrets) value of discrets in base of Lagger polynomials, converting from DT-discrets, start to descend only after fourth, and the velocity of descending is much higher, than in the differential spectrum.

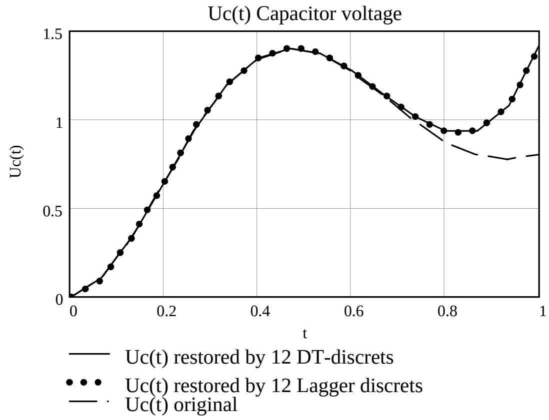

Fig. 5. Restored signal by discrets in base of Lagger polynomials.

The restored signal by 12 discrets of the differential spectrum (Fig. 5, the solid line) and signal obtained by discrets in base of Lagger polynomials (Fig. 5, the line of points) are congruent and MSD is 0.076%, that is the transition in base of Lagger polynomials has taken place with a large degree of accuracy. However such method does not give any advantages. If to attempt to reduce a series (11) at signal recovery, that is, for example, to take 11 Lag-discrets, in this case will be intolerable error of restored signal, in spite of the fact that the value of the last discret more than a hundred times is less previous. It is explained that the value of Lagger polynomials (13) fast grows with increase of discret number, that indemnifies a rapidly decrescent spectrum (Fig. 2).

Thus, the transition in base of orthogonal Lagger polynomials does not give any advantages, and in some cases even it is impossible because of the complex formula of converting (12), that makes impossible precise calculation on modern computer technology.

IV. Transition in base of Legandre polynomials

For transition in base of orthogonal Legandre polynomials the following formula is used:

$$C(i)=(2i+1)\cdot\sum_{j=0}^{j=i}\left(\frac{(-1)^j\cdot(i+j)!}{(j!)^2\cdot(i-j)!}\cdot\sum_{k=0}^{k=km}\left[\frac{X(k-j)}{1+k}\right]\right) \qquad (14)$$

where *km* is maximum known number of discrets of differential spectrum.

Value of Legandre polynomials $T_i(t)$ are evaluated by the following:

$$T_i(t)=\sum_{k=0}^{k=i}\left(\frac{(-1)^k\cdot(i+k)!}{(k!)^2\cdot(i-k)!}\cdot t^k\right) \qquad (15)$$

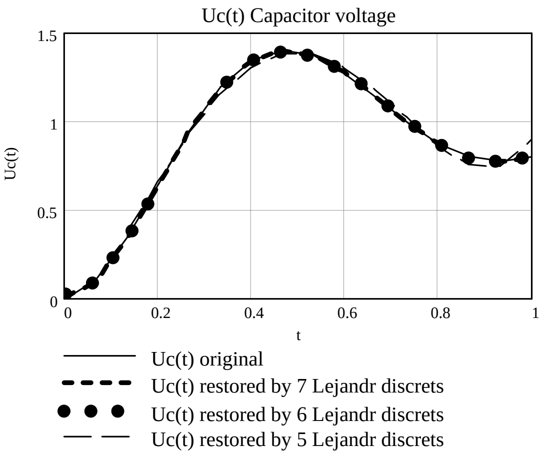

Figure 6 - Restored signal by discrets in base of Legandre polynomials.

As it is visible from the graph (Fig. 2, Legandre discrets), since ninth discrete, the spectrum (L-discrets) of the signal in base of Legandre polynomials monotonically grows, so it is impossible uniquely to say, how much L-discrets will be enough for signal recovery with the adequate accuracy. However for precise restoring of the required signal it is enough to take only 7 L-discrets, and the MSD will make 0.28% (Fig. 6, the bold dash line). If to discard the last term of the series (11), we shall get the graph (Fig. 6, the line of points) with a little larger MSD – 0.56%. At usage only five L-discrets for signal recovery (Fig. 6, the thin dotted line) the MSD makes 3.8%, and the waveform is quite acceptable for a crude approximation.

Thus, the transition in base of orthogonal Legandre polynomials gives essential advantage in cutting number of series terms. In our case length of a series was reduced approximately in 4 times (from 21 up to 5) at the same values MSD.

V. Transition in base of Chebyshev's 1 type polynomials

For transition in base of orthogonal Chebishev's 1 type polynomials will be used the following formula:

$$C(i) = \sum_{k=i}^{k=\left[\frac{km+i}{2}\right]} \left(\frac{X(2k-i)\cdot\rho(i)}{H^{2k-i}\cdot 2^{2k-i-1}} \cdot \frac{(2k-i)!}{k!\cdot(k-i)!} \right),$$

$$\rho(i) = \begin{cases} 1/2, & i = 0 \\ 1, & i \neq 0 \end{cases} \qquad (16)$$

where *km* is maximum known number of discrets of the differential spectrum.

Value of Chebyshev's 1 type polynomials $T_i(t)$ are evaluated as the following:

$$T_i(t) = cos(i \cdot arccos(t)) \qquad (17)$$

In the same way as for Legandre polynomials we shall looking for an approximating polynomial for signal restored by 50 DT-discrets. Starting already from the first discret, the spectrum in base of Chebyshev's 1 type polynomials (C-discrets) monotonically descends (Fig. 2, Chebyshev discrets), so it is possible to attempt to define number, since which the sum following discarded discrets will be minute.

Thus, for precise signal recovery it is enough to take only 13 C-discrets (Fig. 7, the line of points) and MSD makes 0.1%. For the signal restored by 11 C-discrets (Fig. 7, the thick dash line), the MSD is 0.8%, thus the waveform precisely concurs with the original signal (Fig. 7, the solid line). If to take 10 C-discrets, the waveform (Fig. 7, the thin dash line) will be a little distorted, and the MSD will make 2.3%.

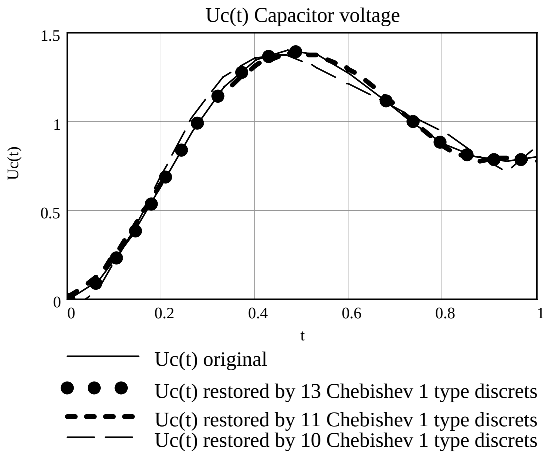

Fig. 7. Restored signal by discrets in base of Chebyshev's 1 type polynomials.

Thus, transition in base of orthogonal Chebyshev's 1 type polynomials gives essential cutting of series terms number. In this case series length was reduced approximately in 2-3 times (from 31 up to 13) at the same approximately values of MSD. Also it is possible beforehand to evaluate an amount indispensable discrets for signal recovery, as all discrets, since the first, monotonically descend (Fig. 2, Chebyshev discrets).

VI. Transition in base of displaced Chebyshev's 1 type polynomials

For transition in base of displaced Chebyshev polynomials the following formula will be used:

$$C(i) = \sum_{k=i}^{km} \left(\frac{X(k)\cdot\rho(i)}{2^{2k-1}} \cdot \frac{(2k)!}{(k+i)!\cdot(k-i)!} \right), \qquad (18)$$

where *km* is maximum known number of discrets of the differential spectrum.

Value of displaced Chebyshev polynomials $T_i(t)$ are evaluated under the following formula:

$$T_i(t) = i \cdot \sum_{j=0}^{i} \left[\frac{(-1)^j \cdot (2i-j-1)!}{j!\cdot(2i-2j)!} \cdot (4t)^{i-j} \right] \qquad (19)$$

As it is visible from the graph (Fig. 2, Displaced Chebyshev discrets), starting already from the first discrete (CS-discrets) the spectrum monotonically (next nearest) descends, and it is possible to attempt to define number, since which the sum following discarded discrets will be minute as for Chebyshev's 1 type polinomials. And it is uneasy to note that the CS-discrets spectrum much faster descends than spectrum of the C-discrets (Fig. 2, Chebyshev's 1 type discrets). It is possible from this to make a conclusion that in base of displaced Chebyshev polynomials the series (11) will be shorter that is discrete number. From which it is possible to discard the stayed terms, will be less that is confirmed by graphs on Fig. 8.

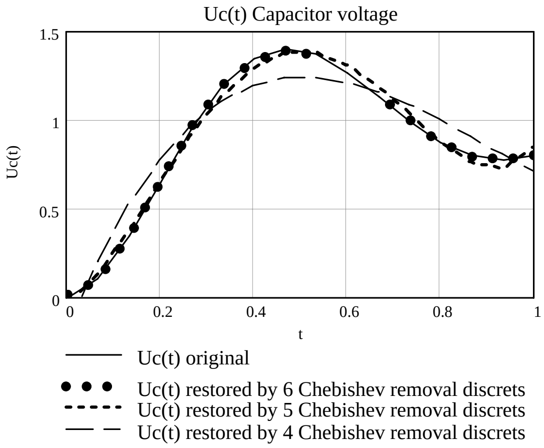

Fig. 8. Restored signal by discrets in base of displaced Chebyshev polynomials.

For precise signal recovery it is enough to take 6 CS-discrets (Fig. 8, the line of points), and MSD makes only 0.09%. For a signal restored by 5 CS-discrets (Fig. 8, the thick dash line), MSD is equal 1.8%, and the waveform differs from the original signal a little. If to take only 4 CS-discrets, the waveform will be very hardly distorted, and MSD will make 3.1%.

Thus, transition in base of displaced Chebyshev's 1 type polynomials gives approximately double cutting of the series as contrasted to previous case, it is explained to that at converting in base of displaced polynomials all initial discrets are used (18), and in the previous case only half (either even, or uneven) (16). In this case length of series was reduced more than in 4 times (from 31 up to 6) at the same approximately values of MSD. Also it is possible beforehand to evaluate an amount indispensable discrets for signal recovery, as all discrets, since the first, monotonically (next nearest) descend (Fig. 2, Displaced Chebyshev discrets).

VII. Conclusion

Table 1. Transition performances in different bases.

Conversion	Amount of series terms after transition / MSD			
DT-conversion	discrets amount	**29**	**25**	**21**
	MSD	0.1%	0.56%	2.6%
Lagger polynomials	discrets amount	**11**	-	-
	MSD	0.08%	-	-
Legandre polynomials	discrets amount	**7**	**6**	**5**
	MSD	0.28%	0.56%	3.8%
Chebyshev polynomials	discrets amount	**13**	**11**	**10**
	MSD	0.10%	0.80%	2.3%
Displaced Chebyshev	discrets amount	**6**	-	**5**
	MSD	0.09%	-	1.8%

In this paper the method for determination of transfer characteristic in circuits with variable parameters in the form of extended series by using Differential-Taylor transformations was considered. Also ways of increasing solution precision at reduction of number of series terms by using conversion from degree in a base of orthogonal polynomials. The orthogonal polynomials of Lagger, Legandre, 1 type and displaced Chebyshev's polynomials were considered. Transition formulas from DT-base in bases of orthogonal polynomials were derived and tested, and also the graphs and signal spectrum were counted and analyzed.

In the table 1 the outcomes of expansion in bases of orthogonal polynomials are shown. From this table it is visible that the best performances have the displaced Chebyshev polynomials, with the help of which the initial series of 29 terms is possible to reduce practically in 5 times – up to 6 terms, and the MSD has not worsened 0.09%. Watching for the graphs of spectrum (Fig. 2) and restored signals (Fig. 7, 8) in expansion on Chebyshev polynomials can roughly be marked, that for precise signal recovery, discrets, which values on absolute value are less the first one in 100…1000 times can be discarded.

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